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AN INVERSE PROBLEM FOR AN ELASTIC MEDIUM CONTAINING A PHYSICALLY NON-LINEAR INCLUSION†

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A plane finite elastic domain containing a physically non-linear inclusion is considered. The problem of determining the loads acting on the outer boundary of the domain that produce a given uniform stress–strain state in the inclusion is formulated and solved. © 2000 Elsevier Science Ltd. All rights reserved.

It is well known [1] that an ellipsoidal elastic inclusion in an elastic space that is subject at infinity to the action of finite stresses will be in a uniform stress–strain state (SSS). This result was extended in [2] to the case of a physically non-linear inclusion in an infinite elastic space. This raises the question, whether one can realize a uniform SSS in an inclusion of arbitrary shape contained in an elastic domain of finite size. This problem will be solved here in a two-dimensional formulation.

1. FORMULATION AND SOLUTION OF THE PROBLEM

Consider an isotropic finite elastic domain S with a physically non-linear inclusion S^*, under conditions of plane deformation or a generalized plane stressed state. The outer boundaries of both the domain and the inclusion are assumed to be simple closed contours, denoted respectively by L and L^* (i.e. the latter separates the elastic medium from the inclusion). Hooke's law is assumed to hold in the domain S; in both cases of the two-dimensional problem, Hooke's law may be written in the form

$$8\mu\varepsilon_{kl} = (\varkappa - 1)\sigma_{nn}\delta_{kl} + 4\sigma^0_{kl}, \quad k, l = 1, 2 \tag{1.1}$$

$$\sigma^0_{kl} = \sigma_{kl} - \tfrac{1}{2}\sigma_{nn}\delta_{kl}$$

where σ^0_{kl} and δ_{kl} are the components of the two-dimensional stress deviator and the unit tensor, μ is the shear modulus, $\varkappa = 3 - 4\nu$ in plane deformation or $\varkappa = (3 - \nu)/(1 + \nu)$ in a generalized plane stressed state (ν is Poisson's ratio) [3]; repeated indices signify summation from 1 to 2. The coordinate system Ox_1x_2 is chosen so that $(0, 0) \in S^*$.

We will write the equations for the inclusion S^* in the following general form

$$\varepsilon^*_{kl} = F_{kl}(\sigma^*_{mn}), \quad k, l, m, n = 1, 2 \tag{1.2}$$

where F_{kl} are, generally speaking, non-linear operators (for example, for a viscoelastic-plastic medium, exhibiting the properties of physically non-linear creep [4]), which in special cases may be functions (linear and non-linear elasticity, plasticity under simple ways of loading, etc.).

We will formulate the main problem, which may be classified as an inverse problem: it is required to determine the external loads on the boundary L so that the SSS thus produced in the inclusion S^* will be a given uniform state (i.e. independent of the coordinates x_k), characterized by components σ^*_{kl} and ε^*_{kl} $(k, l = 1, 2)$ satisfying relations (1.2). It is assumed that the load field produced at the boundary L^*, which we denote by $p_k = \sigma_{kl}n_l$ (where n_k are the components of a unit vector normal to L^*) and the displacements u_k are continuous $(k = 1, 2)$. The problem is considered in a geometrically linear formulation.

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To solve the problem we define in S^* a stress function $U^* = U^*(z, \bar{z})$, where z and $\bar{z}$ are treated as two independent complex variables [5] (U^* may also depend on the time t if $\sigma^*_{kl} = \sigma^*_{kl}(t)$). Since [5]

$$\sigma^*_{11} + \sigma^*_{22} = 4\frac{\partial^2 U^*}{\partial z \partial \bar{z}}, \quad \sigma^*_{22} - \sigma^*_{11} + 2i\sigma^*_{12} = 4\frac{\partial^2 U^*}{\partial z^2}$$

and σ^*_{kl} is independent of x_1 and x_2, and hence also of z and $\bar{z}$, it follows, neglecting terms linear in z and $\bar{z}$ that have no effect on the stressed state, that

$$2U^* = Az\bar{z} + Bz^2/2 + \bar{B}\bar{z}^2/2 \tag{1.3}$$

$$2A = \sigma^*_{11} + \sigma^*_{22}, \quad 2B = \sigma^*_{22} - \sigma^*_{11} + 2i\sigma^*_{12}$$

Since the stresses transmitted across the boundary L^* are continuous, the function $f = 2\partial U/\partial \bar{z}$, introduced in [3] must be continuous on it; therefore, by (1.3),

$$2\partial U/\partial \bar{z} = Az + \bar{B}\bar{z} \quad \text{on} \quad L^* \tag{1.4}$$

where $U = U(z, \bar{z})$ is the stress function for the domain S.

We will express the complex displacement $w^* = u^*_1 + iu^*_2$ in S^* in term of ε^*_{kl}. It follows from the Cauchy relations that [5]

$$2\partial w^*/\partial \bar{z} = \varepsilon^*_{11} - \varepsilon^*_{22} + 2i\varepsilon^*_{12}, \quad \partial w^*/\partial z + \partial \bar{w}^*/\partial \bar{z} = \varepsilon^*_{11} + \varepsilon^*_{22}$$

Hence, taking into account that ε^*_{kl} $(k, l = 1, 2)$ are independent of z and $\bar{z}$ and assuming that $w^*(0, 0) = 0$, we obtain

$$\begin{aligned} &2w^* = Cz + D\bar{z} \\ &C = \varepsilon^*_{11} + \varepsilon^*_{22} + 2i\varepsilon^*, \quad D = \varepsilon^*_{11} - \varepsilon^*_{22} + 2i\varepsilon^*_{12} \end{aligned} \tag{1.5}$$

where ε^* is an arbitrary constant, equal to the value of the "rotation" in S^*.

Consider a conformal mapping of the infinite domain exterior to L^* (and containing S) onto the exterior of the unit circle γ^* of the complex ζ plane, of the form

$$z = \omega(\zeta) = m_1\zeta + \sum_{k=1}^{\infty} m_{-k}\zeta^{-k}, \quad \zeta = \rho e^{i\theta} \tag{1.6}$$

Then relations (1.4)–(1.6) for $\rho = 1$ yield boundary conditions for the functions $\varphi(\zeta)$ and $\psi(\zeta)$ that define the SSS when $[\zeta] > 1$ [3, 5]:

$$\begin{aligned} &\varkappa_k \overline{\varphi(\sigma)} - \overline{\omega(\sigma)}\varphi'(\sigma)/\omega'(\sigma) - \psi(\sigma) = \overline{F_k(\sigma)} \quad (k = 1, 2) \text{ on } \gamma^* \\ &\sigma = e^{i\theta}, \quad \varkappa_1 = -1, \quad \varkappa_2 = \varkappa \\ &\overline{F_1(\sigma)} = -A\overline{\omega(\sigma)} - B\omega(\sigma), \quad \overline{F_2(\sigma)} = \mu[\bar{C}\overline{\omega(\sigma)} + \bar{D}\omega(\sigma)] \end{aligned} \tag{1.7}$$

It follows from (1.7) that

$$(\varkappa + 1)\overline{\varphi(\sigma)} = (A + \mu\bar{C})\overline{\omega(\sigma)} + (B + \mu\bar{D})\omega(\sigma) \text{ on } \gamma^*$$

Hence, using well known methods [3, 5], we obtain

$$\varphi(\zeta) = [(A + \mu C)\omega(\zeta) + (\bar{B} + \mu D)\bar{\omega}(\zeta^{-1})]/(\varkappa + 1), \quad |\zeta| > 1 \tag{1.8}$$

where we have used the equality $\bar{\omega}(\zeta^{-1}) = \omega(\bar{\zeta}^{-1})$ [3] and taken into account that by (1.3) $\bar{A} = A$.

For the known function $\varphi(\zeta)$ we have [5]

$$\begin{aligned} &\psi(\zeta) = -\bar{\varphi}(\zeta^{-1}) - \bar{\omega}(\zeta^{-1})\varphi'(\zeta)/\omega'(\zeta) - \bar{F}_1(\zeta^{-1}), \quad |\zeta| > 1 \\ &\bar{F}_1(\zeta^{-1}) = -A\bar{\omega}(\zeta^{-1}) - B\omega(\zeta) \end{aligned} \tag{1.9}$$

Substituting (1.8) into (1.9), we find that

$$\psi(\zeta)=\{(\varkappa B-\mu\bar{D})\omega(\zeta)+[(\varkappa-1)A-2\mu\,\mathrm{Re}\,C]\omega_1(\zeta)-$$
$$-(\bar{B}+\mu D)\omega_1(\zeta)\omega_1'(\zeta)/\omega'(\zeta)\}/(\varkappa+1),\quad \omega_1(\zeta)\equiv\bar{\omega}(\zeta^{-1}),\quad |\zeta|>1 \tag{1.10}$$

As can be seen from (1.6), (1.8) and (1.10), the functions $\varphi(\zeta)$ and $\psi(\zeta)$ are holomorphic in the ring $1<|\zeta|<R$, where $R^{-1}=\lim|m_{-n}|^{1/n}$ as $n\to\infty$, which is possible if $R>1$. In particular, if the mapping function $\omega(\zeta)$ contains a finite number of terms under the summation sign in (1.6), then $R=\infty$.

To find the required external loads, it will suffice to determine the values of the aforementioned function $f(z,\bar{z})=2\partial U/\partial\bar{z}$ on the boundary L. Putting $f[\omega(\zeta),\overline{\omega(\zeta)}]=\mathrm{F}(\zeta,\bar{\zeta})$ we have [3]

$$F=\varphi(\zeta)+\omega(\zeta)\overline{\varphi'(\zeta)}/\overline{\omega'(\zeta)}+\overline{\psi(\zeta)}$$

Substituting φ and ψ into this equality from (1.8) and (1.10), we obtain

$$\begin{aligned}F(\zeta,\bar{\zeta})=\{[2(A+\mu\,\mathrm{Re}\,C)+(B+\mu\bar{D})\overline{\omega_1'(\zeta)}/\overline{\omega'(\zeta)}](\omega(\zeta)-\overline{\omega_1(\zeta)})+\\+(\bar{B}+\mu D)(\omega_1(\zeta)-\overline{\omega(\zeta)})\}/(\varkappa+1)+A\overline{\omega_1(\zeta)}+\bar{B}\overline{\omega(\zeta)}\end{aligned} \tag{1.11}$$

Let us assume that the contour γ in the ζ plane corresponding to the boundary L of S lies entirely in the ring $1<|\zeta|<R$. Let $z=\Omega(\zeta_1)$, $\zeta_1=\rho_1 e^{i\theta_1}$ be a conformal mapping of the domain $S^*\cup S$ onto the interior (or exterior) of the unit circle of the ζ_1 plane. Then on γ we have

$$\zeta=F_3(\sigma_1),\quad F_3(\sigma_1)\equiv\omega^{-1}[\Omega(\sigma_1)],\quad \sigma_1=e^{i\theta_1}$$

(ω^{-1} denotes the function inverse to ω; ω^{-1} exists, since $\omega'(\zeta)\neq 0$ for $|\zeta|>1$). Substituting this expression into (1.11), we find the value of the function f on the unit circle of the ζ_1 plane to be

$$f_1(\sigma_1)=F[F_3(\sigma_1),\ \overline{F_3(\sigma_1)}]$$

The components of the external loads that must be applied at the boundary L are precisely the components of the stresses $\sigma_{\rho 1}$ and $\sigma_{\rho\theta 1}$ in the curvilinear coordinates (ρ_1,θ_1) associated with the above-mentioned conformal mapping, for $\rho_1=1$. These stresses are defined as follows [5]:

$$\sigma_{\rho 1}+i\sigma_{\rho\theta 1}=f_1'(\sigma_1)/\Omega'(\sigma_1) \tag{1.12}$$

Note that the functions φ and ψ of (1.8) and (1.10) that define the SSS for $|\zeta|>1$ are expressed solely in terms of the mapping function ω (for given components σ_{kl}^* and ε_{kl}^* in S^*) associated with the form of the boundary L^*. The outer boundary L has no influence whatever on the SSS; the required loads on it are determined by its geometry and by the already determined functions φ and ψ. In this sense, the solution may also be continued beyond L, provided that the values of $|\zeta|$ lie in the ring $1<|\zeta|<R$.

2. THE UNIQUENESS OF THE SOLUTION OF THE PROBLEM

As can be seen from formulae (1.8) and (1.10), given the SSS on S^* (that is, given σ_{kl}^* and ε_{kl}^*), and provided that the contour γ corresponding to L lies in the ring $1<|\zeta|<R$, a solution for the SSS in S exists and is unique. This follows from the fact that the functions $\varphi(\zeta)$ and $\psi(\zeta)$ of (1.8) and (1.10) do not contain logarithmic terms and the arbitrary constant $2\varepsilon^*\equiv\mathrm{Im}\,C$ occurring in (1.8) has no effect on the SSS. There will be no arbitrariness in the definition of $\varphi(\zeta)$, if the magnitude of the rotation ε^* is given in S^*. By *(1.12), the SSS thus determined in the domain S uniquely defines the required loads on the outer contour L.

We will now show that the converse is also true: given the loads on L as in (1.12), and assuming that certain restrictions are imposed on Eqs (1.2), the SSS thus realized in the domain $S^*\cup S$ is precisely that considered above, that is, the inclusion will be under conditions of a uniform SSS.

In the interests of greater clarity, let us make Eqs (1.2) somewhat more specific, while still leaving them fairly general [6]:

$$\varepsilon_{kl}^*=a_{klmn}^*\sigma_{mn}^*+\varepsilon_{kl}^{*N},\quad k,l=1,2 \tag{2.1}$$

Indeed, if $\varepsilon_{kl}^{*N} = 0$, the inclusion is linearly elastic; in particular, if it is isotropic, Eqs (2.1) take the form of (1.1) (with specific constants $\varkappa^*$ and μ^*). If $a_{klmn}^* = 0$, but $\varepsilon_{kl}^{*N} = \varepsilon_{kl}^{*N} = (\sigma_{mn}^*)$ are known functions, the inclusion is non-linearly elastic or subject to the deformation theory of plasticity. In the general case ($a_{klmn}^* \neq 0$), ε_{kl}^{*N} are irreversible deformations (plastic, viscous, creep deformations, or sums of all these deformations).

The above-mentioned restrictions correspond to standard assumptions on the stability of the deformation process, which, for a linearly or non-linearly elastic inclusion, when $\varepsilon_{kl}^{*N} = \varepsilon_{kl}^{*N} = (\sigma_{mn}^*)$, reduce to the following relations (where Δ denotes an increment) [4]

$$a_{klmn}^*\Delta\sigma_{kl}^*\Delta\sigma_{mn}^* > 0, \quad \Delta\varepsilon_{kl}^{*N}\Delta\sigma_{kl}^* > 0 \quad \text{if} \quad \Delta\sigma_{kl}^*\Delta\sigma_{kl}^* \neq 0 \tag{2.2}$$

In the case when ε_{kl}^{*N} are irreversible deformations, the analogous condition is [4, 6]

$$\int_0^t \Delta\dot{\varepsilon}_{kl}^{*N}\Delta\sigma_{kl}^* dt > 0 \tag{2.3}$$

where t is the time (for viscous media) or the load parameter (for plastic media).

Inequalities (2.2) and (2.3) become equalities only if $\Delta\sigma_{kl}^* = 0$ ($k, l = 1, 2$).

With these loads on L and under conditions (2.2) or (2.3), the SSS in $S^* \cup S$ is uniquely defined.

Indeed, let us assume that, besides the solution just determined, another SSS exists; denote the differences between corresponding quantities by the symbol Δ. Then, by the equation of virtual work and the continuity of the loads and displacements on L^*, we have [4, 7]

$$\int_S \Delta\varepsilon_{kl}\Delta\sigma_{kl} dS + \int_{S^*} \Delta\varepsilon_{kl}^*\Delta\sigma_{kl}^* dS = 0 \tag{2.4}$$

Substituting Eqs (1.1) and (2.1) into (2.4) and taking (2.2) into account, we obtain $\Delta\sigma_{kl} = 0$ in S and $\Delta\sigma_{kl}^* = 0$ in S^* ($k, l = 1, 2$).

If the stability condition is (2.3), we can replace Eq. (2.4) by a similar one, with the strain increments $\Delta\varepsilon_{kl}$ and $\Delta\varepsilon_{kl}^*$ replaced by their rates. After integrating the resulting equation with respect to time from zero to t and using (2.3) and the equalities

$$\Delta\sigma_{kl}\big|_{t=0} = \Delta\sigma_{kl}^*\big|_{t=0} = 0$$

which follow from the uniqueness of the solution corresponding to time $t = 0$ of the elastic or viscoelastic problems (of which we spoke above), we find, by analogy with a previous publication [4], that $\Delta\sigma_{kl} = 0$ in S and $\Delta\sigma_{kl}^* = 0$ in S^* for $t > 0$.

3. EXAMPLES

Using well-known relations for the stress components in the curvilinear coordinates (ρ, θ) associated with the conformal mapping $z = \omega(\zeta)$ [3],

$$\sigma_\rho + \sigma_\theta = 2[\Phi(\zeta) + \overline{\Phi(\zeta)}]$$

$$\sigma_\theta - \sigma_\rho + 2i\sigma_{\rho\theta} = 2\zeta^2\rho^{-2}[\overline{\omega(\zeta)}\Phi'(\zeta) + \psi'(\zeta)]/\overline{\omega'(\zeta)}$$

$$\Phi(\zeta) = \varphi'(\zeta)/\omega'(\zeta)$$

and substituting the functions $\varphi(\zeta)$ and $\psi(\zeta)$ from (1.8) and (1.10) into these relations, we obtain ($|\zeta| > 1$)

$$\begin{aligned}
&\sigma_\rho + \sigma_\theta = 4\operatorname{Re}[A + \mu C + (\overline{B} + \mu D)\omega_2(\zeta)]/(\varkappa + 1)\\
&\sigma_\theta - \sigma_\rho + 2i\sigma_{\rho\theta} = 2e^{2i\theta}\{(\overline{B} + \mu D)[\overline{\omega(\zeta)}\omega_2'(\zeta) - (\omega_1(\zeta)\omega_2(\zeta))'] +\\
&+(\varkappa B - \mu\overline{D})\omega'(\zeta) + [(\varkappa - 1)A - 2\mu\operatorname{Re}C]\omega_1'(\zeta)\}/[(\varkappa + 1)\overline{\omega'(\zeta)}]\\
&\omega_2(\zeta) \equiv \omega_1'(\zeta)/\omega'(\zeta)
\end{aligned} \tag{3.1}$$

(the function $\omega_1(\zeta)$ was defined in (1.10)).

As can be seen from (3.1) and (1.6), the stresses will remain bounded as $|\zeta| \to \infty$ only if the mapping function (1.6) contains only the first two non-zero terms, that is, $m_{-k} = 0$ for $k \geqslant 2$, which corresponds to an elliptical inclusion. Thus, if the contour L^* is an ellipse, the solution constructed in Section 1 for the SSS in S may be continued beyond L, including an infinitely distant point

As an example, consider the more general situation in which

$$\omega(\zeta) = R_0(\zeta + m\zeta^{-n}), \quad 0 < mn < 1, \quad n \geqslant 1 \tag{3.2}$$

where the constants R_0 and m may be considered to be real, and moreover $m > 0$ (this may always be ensured by rotating the coordinate axes in the z and ζ planes); n is a natural number. For $n = 1$ the function (3.2) is identical with the above-mentioned mapping function of the exterior of the ellipse onto the exterior of the unit circle γ^* in the ζ plane.

Substituting (3.2) into (3.1), we obtain

$$\begin{aligned}
&(\varkappa+1)(\sigma_\rho+\sigma_\theta) = 4\,\mathrm{Re}[A+\mu C+(\bar{B}+\mu D)f_2(\zeta)/f_3(\zeta)]\\
&(\varkappa+1)(\sigma_\theta-\sigma_\rho+2i\sigma_{\rho\theta}) = 2\sigma^2\{[(\varkappa-1)A-2\mu\,\mathrm{Re}\,C]f_2(\zeta)+\\
&+(\varkappa B-\mu\bar{D})f_3(\zeta)+(\bar{B}+\mu D)[(\bar{\zeta}+m\bar{\zeta}^{-n}-\zeta^{-1}-m\zeta^{n})(2(1-m^2n^3)\zeta^{-3}+\\
&+mn(n-1)(\zeta^{n-2}+\zeta^{-n-4}))-f_2^2(\zeta)f_3(\zeta)]f_3^{-2}(\zeta)\}/\overline{f_3(\zeta)}\\
&f_2(\zeta) = -\zeta^{-2}+mn\zeta^{n-1}, \quad f_3(\zeta) = 1-mn\zeta^{-n-1}, \quad \sigma = e^{i\theta}
\end{aligned} \tag{3.3}$$

Hence it follows that for $n > 1$ the principal terms determining the behaviour of the quantities indicated in (3.3) as $|\zeta| \to \infty$ have the form

$$\begin{aligned}
&(\varkappa+1)(\sigma_\rho+\sigma_\theta) \sim 4mn\,\mathrm{Re}(\bar{B}+\mu D)\zeta^{n-1}\\
&(\varkappa+1)(\sigma_\theta-\sigma_\rho+2i\sigma_{\rho\theta}) \sim -2m^2n(2n-1)(\bar{B}+\mu D)\sigma^2\zeta^{2n-2}
\end{aligned}$$

Suppose the contour γ in the ζ plane corresponding to the boundary L is a circle of radius ρ_0, that is, the mapping mentioned at the end of Section 1 has the form

$$\Omega(\zeta_1) = \omega(\rho_0\zeta_1), \quad \zeta_1 = \rho_0^{-1}\zeta$$

Then it is easy to derive from (3.3) an expression for $\sigma_\rho - i\sigma_{\rho\theta}$ when $\zeta = \rho_0\sigma$, which determines the required external loads on L as functions of σ.

For $n = 1$ (an elliptical inclusion), well-known results [3] applied to (1.8), (1.10) and (3.2) yield the following relations between the stresses σ_{kl}^∞ and the rotation ε^∞ at infinity, on the one hand, and the analogous quantities in S^*, on the other

$$\begin{aligned}
&(\varkappa+1)\Gamma = A+\mu C+m(\bar{B}+\mu D)\\
&(\varkappa+1)\Gamma' = \varkappa B-\mu\bar{D}+m[(\varkappa-1)A-2\mu\,\mathrm{Re}\,C]-m^2(\bar{B}+\mu D)\\
&4\Gamma \equiv \sigma_{11}^\infty+\sigma_{22}^\infty+8i\mu\varepsilon^\infty/(\varkappa+1), \quad 2\Gamma' \equiv \sigma_{22}^\infty-\sigma_{11}^\infty+2i\sigma_{12}^\infty
\end{aligned} \tag{3.4}$$

It can be shown that, if stability conditions (2.2) (or (2.3)) are satisfied, then relations (3.4) and (2.1) are uniquely solvable as equations in σ_{kl}^* ($k, l = 1, 2$) and ε^*. Moreover, given the quantities Γ and Γ' in (2.2) or (2.3), it follows that the SSS in the domain $S^* \cup S$ is uniquely defined, that is, under the action of these forces at infinity, the inclusion is subject to a uniform SSS as defined by (3.4) and (2.1).

The proof of this fact is analogous to that presented in Section 2 and is as follows. Considering instead of S a finite elastic domain S_r bounded by an outer contour L_r which is a circle of radius r, we obtain the following expression for the difference of two possible solutions [4, 7]

$$\int_{S_r}\Delta\varepsilon_{kl}\Delta\sigma_{kl}dS+\int_{S^*}\Delta\varepsilon_{kl}^*\Delta\sigma_{kl}^*dS = I, \quad I \equiv \int_{L_r}\Delta u_k\Delta p_k ds$$

Letting r tend to infinity and taking into account that $I \to 0$ as $r \to \infty$ [3], we obtain (2.4), which is possible only if $\Delta\sigma_{kl} = 0$ in S and $\Delta\sigma_{kl}^* = 0$ in S^* ($k, l = 1, 2$).

4. CONCLUDING REMARKS

As can be seen from the results obtained above, the inverse problem we have been considering has essentially been reduced to a problem of elasticity theory for a doubly connected domain S, with both displacements and loads given on the inner boundary L^*, but with no conditions specified on the outer boundary L (these conditions have to be determined). Such a problem has been studied before [8], and given the name of "the (**u**, **p**) problem". In the two-dimensional case, use has also been made [8] of boundary conditions of the form (1.7) with arbitrary functions $F_k(\sigma)(k = 1, 2)$(that is, not necessarily corresponding to a uniform SSS in S^*), which at once imply the following boundary condition on L^* for $\varphi(\zeta)$

$$(\varkappa + 1)\varphi(\sigma) = F_2(\sigma) - F_1(\sigma)$$

This reduces the problem to a well known problem [3]: to determine the functions $\varphi(\zeta)$ and $\psi(\zeta)$ for $|\zeta| > 1$ on the basis of their boundary values at $|\zeta| = 1$.

In this sense, it is obvious that the problem considered in this paper may be generalized as follows: it is to required to choose loads on L in such a way as to produce in S^* a required (not necessarily uniform) SSS satisfying relations (1.2) (or, more specifically, (2.1)) and the usual equations for the equilibrium and compatibility of strains. This problem clearly reduces to "splicing" the SSSs in S^* and S along the boundary L^*, assuming that the stresses and strains satisfy the continuity conditions at L^*. The solution of this problem is unique in the same sense as in Section 2.

It is also obvious that the problem may be extended to the three-dimensional case, in which proper choice of the external loads must produce a given (uniform or non-uniform) SSS in a physically non-linear inclusion (PNLI). If σ^*_{kl} and ε^*_{kl} $(k, l = 1, 2)$ are known, the problem again reduces to the (**u**, **p**) problem, whose solution in the domain S is known to be unique [3, 8]. And conversely, as in Section 2, given the external loads, the SSS in the three-dimensional domain $S^* \cup S$ is uniquely defined, as immediately follows from relations (2.1)–(2.4). In particular, in the case of an infinite domain S containing an ellipsoidal PNLI, a uniform SSS will be realized in the inclusion, provided that the stresses σ^∞_{kl} $(k, l = 1, 2)$ are finite [2]. Methods have been developed for solving the three-dimensional (**u**, **p**) problem [8, 9].

The problem considered here, of creating a prescribed uniform SSS in a PNLI, may find application in connection with the problem of the optimal fracture of materials and elements of structures and buildings. This problem for a PNLI may be formulated as follows: for what external loads on L will the whole of S^* be fractured simultaneously – instantaneously (assuming elastic-plastic deformation) or in a given time (assuming viscoelastic-plastic deformation or creep) with minimum energy consumption? Hote that such optimal paths of deformation and fracture have been outlined for brittle and viscous materials under conditions of creep and uniform SSS [10].

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